Testing Laplace-Lagrange Theory with N-Body Simulations of Eccentric Protoplanetary Disks

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Introduction

Laplace-Lagrange secular theory models the long-term, averaged orbital evolution of planetary systems. Lithwick & Chiang (forthcoming) show that protoplanetary disks match the theory's eccentricity predictions only if their outer edges are sharp–a condition unlikely in reality, where edges are gradual.

We compare theory to N-body simulations in REBOUND, first verifying agreement in the sharp-edge case, then exploring tapered edges. This work identifies when REBOUND can be trusted and examines how disk-edge structure shapes eccentricity profiles.

Methods

We implement a linear secular solver following Lithwick & Wu (2013) and obtain the mode precession rates and relative eccentricities determined by the particle masses, semi-major axes, and central stellar mass.

We consider three scaled surface density profiles: a sharp-edge, a tapered-edge, and extended tapered-edge. The particles are distributed between 0.9 and 1.1 AU, with various total disk masses and initial eccentricities set to the zero-node eigenmode. The disk is assumed to be pressureless and has no inclination.

We use the WHFAST integrator, and initialize simulations with 30 particles. Gravitational softening lengths were set to the mean Hill radius. We monitor the root-mean-square eccentricity (e) and mean longitude of pericenter (ϖ) to quantify derivations from the initial mode.

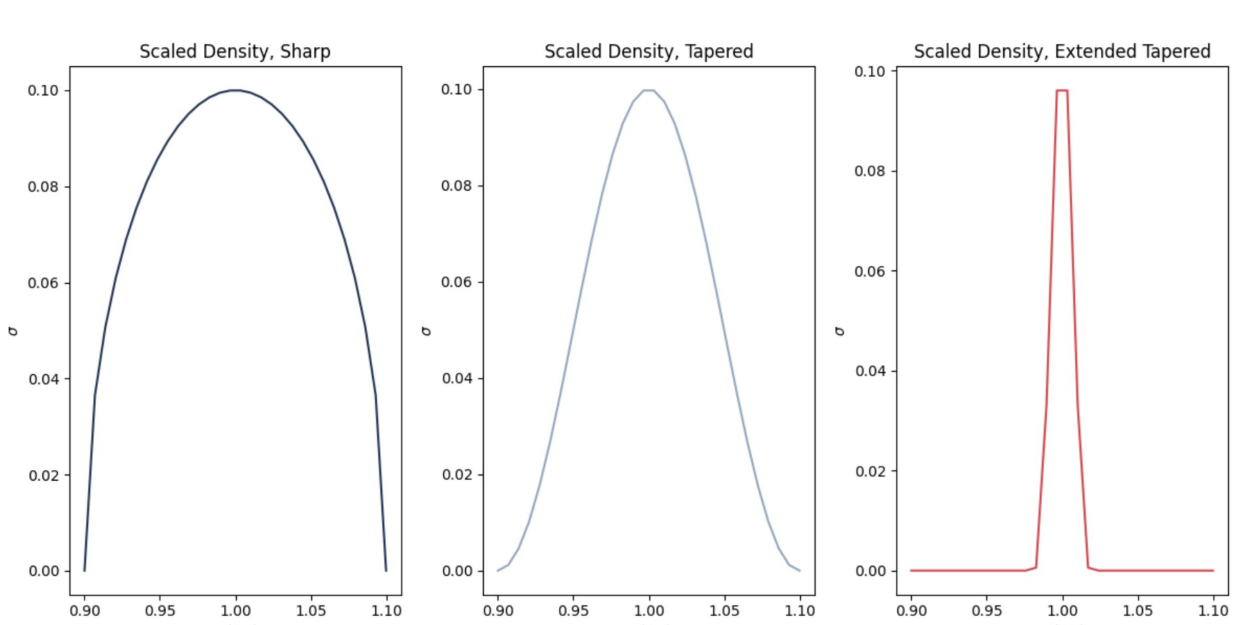

Figure 1: Scaled surface density σ profiles of simulated disks.

Results

Sharp disk edges preserve secular modes best, but lowering the disk mass can offset even poor edge profiles. We tested the three edge profiles across 3 disk mass scalings ($10^{-6}M_*$, $10^{-7}M_*$, and $10^{-8}M_*$).

e_{rms} growth over time

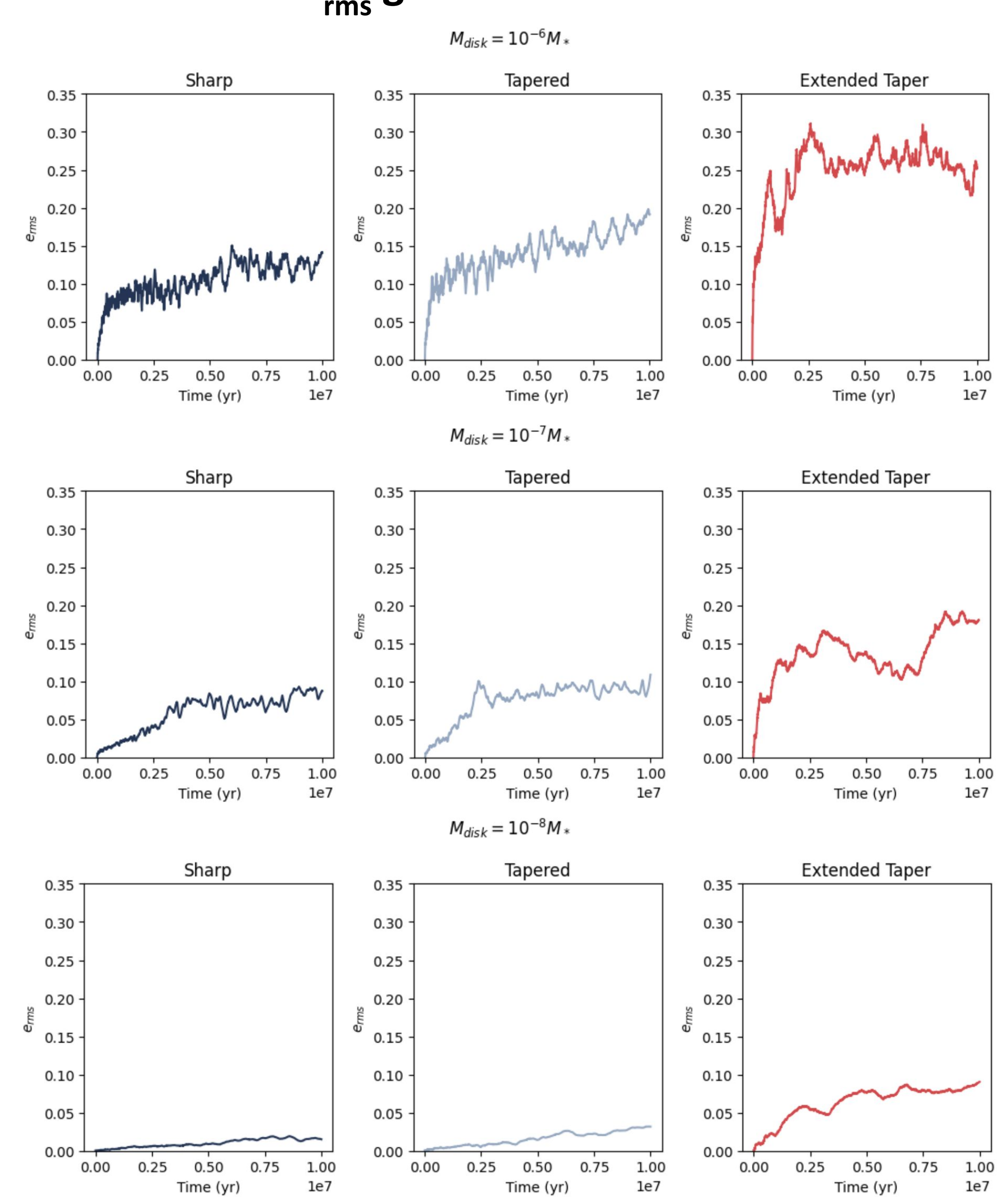

Figure 2: Growth of eccentricity RMS for three edge profiles across disk masses. Sharp edges most faithfully follow the expected secular trend, while tapered edges introduce faster eccentricity growth. At high disk masses, all profiles experience fast eccentricity growth. Extended Taper produces ejections (e ≥ 1). At intermediate mass, Sharp and Tapered maintain coherence somewhat longer, while the Extended Taper still fails early. At low mass, all profiles remain closer to the expected secular mode.

Mean ϖ growth over time

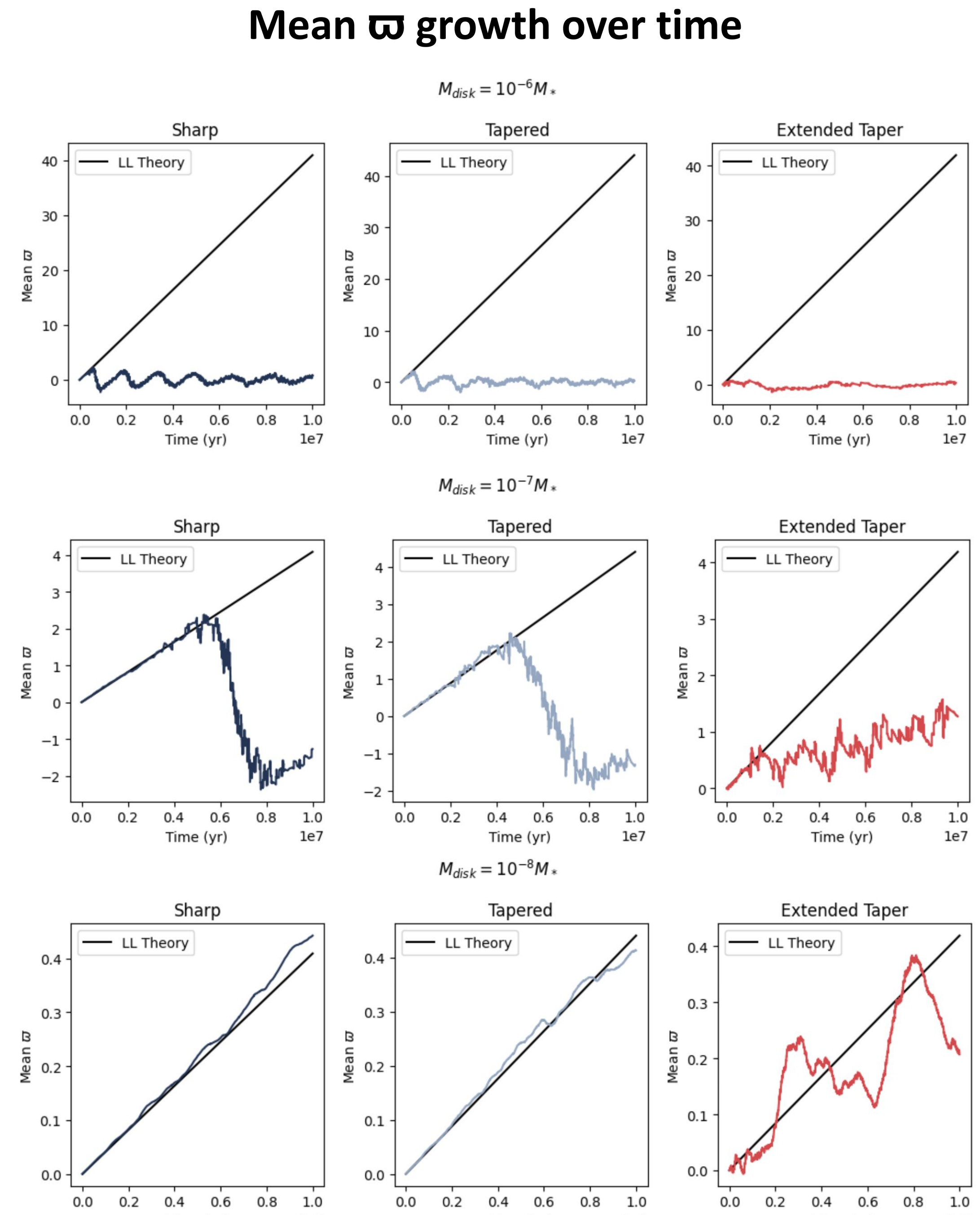

Figure 3: Evolution of mean ϖ for edge profiles across disk masses. Black line represents the expected precession rate as calculated by Laplace-Lagrange (LL) theory. At high disk masses, all profiles diverge rapidly. At intermediate disk masses, simulations maintain coherence for longer, but eventually diverge prematurely. At lower masses, all profiles approximately maintain the initial mode for the duration of the simulation.

Discussion

Two consistent trends emerge:

1. **Edge sharpness matters.** Sharp edges allow the system to maintain the expected secular mode most faithfully. Tapered edges perform moderately well but introduce additional noise and earlier divergence. Extended Tapered edges consistently perform the worst, with discontinuities, premature divergence, and in extreme cases ($10^{-6}M_*$ and $10^{-7}M_*$,) full ejections.
2. **Mass scaling matters even more.** Independent of edge profiles, lower-mass disks maintain coherence longer. At $10^{-8}M_*$, even the poorly performing edge profiles can approximate the expected mode reasonably well for a significant duration, whereas at $10^{-6}M_*$divergence and excitation are immediate and severe.

These results suggest that REBOUND simulations reproduce Laplace-Lagrange secular modes most reliably in low-mass disks with sharp boundaries. Tapered edges destabilize the secular solution, but these effects can be offset by reducing the overall disk mass, effectively prolonging the survival of the coherent mode.

Future Work

- Quantify the dependence of model lifetime on disk edge steepness and total disk mass
- Quantify the dependence of model lifetime on disk width
- Using the established parameters for mode coherence, explore how various tapered edges evolve over time in comparison to sharper edges.
- Develop analytical estimates for leakage rates in discrete systems

M_{disk}	Sharp	Tapered	Extended
$10^{-6}M_*$	6.35e+05	5.84e+05	2.26e+05
$10^{-7}M_*$	6.04e+06	5.31e+06	3.65e+06
$10^{-8}M_*$	Never	Never	Never

Figure 4: Time in years of divergence from theory for various profiles. Divergence calculated as when difference between expected ϖ and simulation ϖ exceeds π/3. As mass of disks decreases, simulation preserves mode for longer. At $10^{-6}M_*$, Sharp remains coherent longest while Extended divergences almost immediately. At $10^{-8}M_*$, no disk experiences divergence within the simulations' time frames.

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